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Equivalence of Graph Signal Processing Using a Hermitian Graph Laplacian and Its Corresponding Graph Laplacian with Duplicated Nodes

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***Abstract*—In our previous work, we introduced a novel graph Laplacian for directed graphs, referred to as the Hermitian graph Laplacian, which enables a unitary graph Fourier transform (GFT) for signals defined on directed graphs. We also showed that the Hermitian graph Laplacian can be transformed into a double-sized real symmetric graph Laplacian, corresponding to an undirected graph with duplicated nodes, which maintains a one-to-one correspondence with the original directed graph. However, the relationship between the GFTs derived from these two Laplacians, as well as the equivalence of their respective graph signal processing (GSP) frameworks, has not yet been thoroughly investigated. In this paper, we establish the equivalence between the two GFTs and that between the corresponding GSP operations. This result suggests that GSP on directed graphs can be fully reduced to GSP on undirected graphs.**

I. INTRODUCTION

In the field of graph signal processing, the graph Fourier transform (GFT) plays a crucial role. It is defined as the basis transformation using the eigenvectors of the so-called graph Laplacian matrix, which reflects the structure of a given graph (see [1], [2], for instance). For an undirected graph, the corresponding graph Laplacian is symmetric, making the GFT an orthogonal transformation. In contrast, for a directed graph, the graph Laplacian is no longer symmetric, which may result in a non-orthogonal GFT, or in some cases, the GFT may not be well-defined due to a non-diagonal Jordan canonical form. To address this issue, several alternative definitions of graph Laplacians and GFTs have been proposed, utilizing techniques including the Jordan canonical form and singular value decomposition (see [3]–[6], for instance). On the other hand, an alternative graph Laplacian, based on transforming an asymmetric adjacency matrix into a Hermitian matrix, has been proposed in [7]–[9], which is the focus of analysis in this paper [1]. Sugie et al. [7] and Muramatsu et al. [8] independently proposed the Hermitian graph Laplacian based on the Hermitian adjacency matrix whose real and imaginary parts consist of the symmetric and skew-symmetric components of an original asymmetric adjacency matrix, which makes the GFT a unitary transformation, similar to that for undirected graphs.

On the basis of the one-to-one correspondence between a complex number and the specific 2×2 real matrix, Kamada et al. [9] pointed out that the Hermitian graph Laplacian of a directed graph can be transformed into a double-sized real symmetric matrix, corresponding to the graph Laplacian of a certain undirected graph with duplicated nodes, which in turn represents the original directed graph. However, the relationship between the GFT based on the Hermitian graph Laplacian and that based on the double-sized symmetric graph Laplacian, corresponding to the undirected graph with duplicated nodes, has not yet been thoroughly investigated.

In this paper, we demonstrate that the two GFTs described above are essentially equivalent by analyzing their eigenstructures. Consequently, signal processing procedures based on these GFTs are also essentially equivalent. This finding implies that graph signal processing for directed graphs can be fully reduced to that for undirected graphs, allowing a wide range of existing knowledge and techniques developed for undirected graphs to be directly applied to directed graphs.

II. GRAPH SIGNAL PROCESSING FOR UNDIRECTED GRAPHS

In this section, we briefly review graph signal processing based on the graph Fourier transform (GFT) for undirected graphs [1], [2].

Let $\mathcal{G} := \{\mathcal{V}, \mathcal{E}\}$ be a given graph, where $\mathcal{V}$ and $\mathcal{E}$ stand for the set of vertices (the set of node numbers) and the set of edges (the set of pairs of node numbers), respectively. In this paper, we assume that $\mathcal{G}$ has no isolated nodes and no self-loops. We also assume $|\mathcal{G}| = N$. We define the adjacency matrix $A = (a_{mn})$ of $\mathcal{G}$ as

$$a_{mn} := \left\{ \begin{array}{ccc} w_{mn} & : & (m,n) \in \mathcal{E} \\ 0 & : & (m,n) \notin \mathcal{E} \end{array} \right. , \tag{1}$$

where $w_{mn} \in \mathbf{R}$ denotes the weight for the edge $(m,n) \in \mathcal{E}$ [2]. Note that since $\mathcal{G}$ is an undirected graph, $a_{mn} = a_{nm}$ holds. Also, $a_{nn} = 0$ due to the assumption that $\mathcal{G}$ has no self-loops. Accordingly, A must be a symmetric matrix with zero diagonal

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[1] Furutani et al. [10] proposed a different way of forming a Hermitian Laplacian. Kitamura et al. [11] proposed a similar framework for filtering on directed graphs, called Augmented GFT. However, these frameworks are out of scope of this paper.

[2] Generally, $w_{mn} > 0$ is assumed. However, we permit negative w_{mn} for the following discussion.

entries. The degree matrix of $\mathcal{G}$ is the diagonal matrix defined as

$$D := (d_{mm}), \quad d_{mm} := \sum_{k=1}^{N} |a_{mk}|. \tag{2}$$

Note that D is positive definite due to the assumption that $\mathcal{G}$ has no isolated nodes. Under these preparations, the so-called graph Laplacian matrix is defined as

$$L := D - A, \tag{3}$$

and its normalized version is defined as

$$\begin{aligned}\mathcal{L} &:= D^{-1/2} L D^{-1/2}\\ &= D^{-1/2}(D-A)D^{-1/2} = I_N - D^{-1/2}AD^{-1/2},\end{aligned} \tag{4}$$

where I_N stands for the identity matrix of degree N. It is well known that the eigenvalues of $\mathcal{L}$ fall into the interval $[0, 2]$ [2].

Let $\mathcal{L} = U\Lambda U'$ be the eigenvalue decomposition of $\mathcal{L}$, where Λ and U denote the diagonal matrix consisting of eigenvalues of $\mathcal{L}$ and the orthogonal matrix consisting of the unit eigenvectors of $\mathcal{L}$, with U' denoting the transpose of U. Now, we assume that each node of $\mathcal{G}$ has a certain value, and let $\boldsymbol{f} \in \mathbf{R}^N$ be the signal vector consisting of those values on the nodes arranged in order of node number. The GFT and its inverse are defined as

$$\boldsymbol{f}_G := U'\boldsymbol{f}, \tag{5}$$

$$\boldsymbol{f} := U\boldsymbol{f}_G. \tag{6}$$

Let $\mathcal{D}$ be the set of $N \times N$ real diagonal matrices. Generally, graph signal processing is formulated by

$$\hat{\boldsymbol{f}} = U\Phi(\Lambda)U'\boldsymbol{f}, \tag{7}$$

where $\Phi(\cdot)$ is a mapping defined on $\mathcal{D}$ that produces an $N \times N$ matrix, which represents a required graph signal processing operation in the graph Fourier domain.

III. HERMITIAN GRAPH LAPLACIANS AND ITS INTERPRETATION

In this section, we briefly review the Hermitian graph Laplacian [7]–[9] and its interpretation [9].

When $\mathcal{G}$ is a directed graph, the corresponding adjacency matrix A is no longer symmetric. Therefore, the eigenvectors of the graph Laplacian may not be orthogonal to each other. Moreover, they may fail to span the whole space due to a non-diagonal Jordan canonical form of $\mathcal{L}$. In order to resolve this issue, we introduced the so-called Hermitian graph Laplacian [7], [9][3], which can be constructed as follows.

The adjacency matrix A of $\mathcal{G}$ can be decomposed into the symmetric and skew-symmetric parts as

$$A := \frac{A + A'}{2} + \frac{A - A'}{2}. \tag{8}$$

Then, its Hermitian version is obtained by multiplying the skew-symmetric part by the imaginary unit i as

$$H := \frac{A + A'}{2} + i\frac{A - A'}{2}, \tag{9}$$

which is trivially Hermitian.

For this $H := (h_{ij})$, Muramatsu et al. [8] introduced a degree matrix defined by

$$D_M := (d_{mm}^{(M)}), \ d_{mm}^{(M)} := \sum_{k=1}^{N} \sqrt{h_{mk}h_{km}}, \tag{10}$$

and gave the following theorem.

Theorem 1: [8] The minimum eigenvalue of the Hermitian graph Laplacian

$$L_M := D_M - H \tag{11}$$

is non-negative.

On the other hand, the normalized version of the Hermitian graph Laplacian (11) was not discussed in [8]. To address this issue, Kamada et al. gave the following theorem in [9].

Theorem 2: [9] The eigenvalues of the normalized Hermitian graph Laplacian

$$\mathcal{L}_M := D_M^{-1/2} L_M D_M^{-1/2} \tag{12}$$

fall into the interval $[0, 2]$.

Kamada et al.[9] also introduced an alternative degree matrix defined by

$$D_K := (d_{mm}^{(K)}), \ d_{mm}^{(K)} := \sum_{k=1}^{N} (|\mathrm{Re}(h_{mk})| + |\mathrm{Im}(h_{mk})|). \tag{13}$$

Let $L_K := D_K - H$ be the Hermitian graph Laplacian based on the degree matrix D_K. Kamada et al. also gave the following theorem.

Theorem 3: [9] The eigenvalues of the normalized Hermitian graph Laplacian

$$\mathcal{L}_K := D_K^{-1/2} L_K D_M^{-1/2} \tag{14}$$

fall into the interval $[0, 2]$.

Since L_M, $\mathcal{L}_M$, L_K and $\mathcal{L}_K$ are Hermitian, their eigenvectors yield a unitary GFT, similar to the cases for undirected graphs. For instance, let $\mathcal{L}_K = U\Lambda U^*$ be the eigenvalue decomposition of $\mathcal{L}_K$, where U^* stands for the Hermitian transpose of U. Then, the GFT and its inverse based on $\mathcal{L}_K$ are reduced to

$$\boldsymbol{f}_G := U^*\boldsymbol{f}, \tag{15}$$

$$\boldsymbol{f} := U\boldsymbol{f}_G, \tag{16}$$

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[3] Muramatsu et al. independently introduced a similar graph Laplacian in [8].

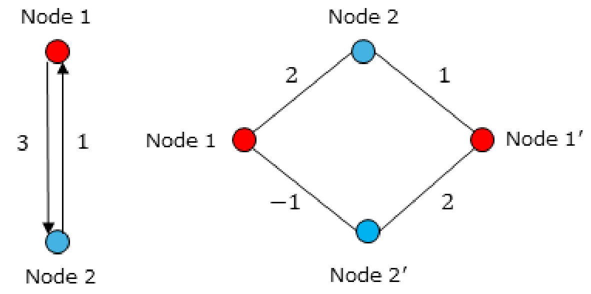

Fig. 1. (a) An original directed graph (left). / (b) The corresponding undirected graph with dualized nodes (right).

since U is unitary. Thus, graph signal processing using a Hermitian graph Laplacian $\mathcal{L}_K$ is reduced to

$$\hat{\boldsymbol{f}} = U\Phi(\Lambda)U^*\boldsymbol{f}. \tag{17}$$

Note that when $\mathcal{G}$ is an undirected graph, the (normalized) Hermitian graph Laplacians based on both D_M and D_K are equivalent to the (normalized) graph Laplacian (3) and (4), which implies that both formulations are natural generalizations of the ordinary graph Laplacian corresponding to an undirected graph. Also note that the result of signal processing (17) may be a complex vector. Therefore, an appropriate interpretation for the result is needed, which is still an open problem.

Hereafter, we do not consider normalization in the following discussion since the range of eigenvalues is not a point at issue of this paper[4].

It is trivial that a complex number $z := x + iy$ has a one-to-one correspondence to the 2×2 matrix

$$Z := \left[\begin{array}{cc} x & -y \\ y & x \end{array}\right]. \tag{18}$$

Moreover, it is well known that the sum, product, and inverse of complex numbers are equivalent to the corresponding matrix operations for (18). It is also trivial that the conjugate $\overline{z}$ corresponds to Z'. Accordingly, by replacing each component in a Hermitian adjacency matrix H with their corresponding 2×2 real matrix, we obtain a real symmetric matrix $\tilde{A}$ twice as large as H, which must correspond to the ordinary adjacency matrix of a certain undirected graph with duplicated nodes [9].

For instance, the adjacency matrix of the directed graph shown in Fig.1(a) is given as

$$A = \left[\begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array}\right], \tag{19}$$

and its Hermitian version is given as

$$H = \left[\begin{array}{cc} 0 & 2+i \\ 2-i & 0 \end{array}\right]. \tag{20}$$

By replacing each component of H into the corresponding 2×2 real matrix (18), we obtain the real symmetric matrix

$$\tilde{A} = \left[\begin{array}{cccc} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{array}\right], \tag{21}$$

which can be regarded as the adjacency matrix corresponding to the undirected graph with duplicated nodes shown in Fig.1(b) in which nodes are arranged in $\mathcal{V} = \{1, 1', 2, 2'\}$.

The degree matrix D_K of H is given as

$$D_K = \left[\begin{array}{cc} 3 & 0 \\ 0 & 3 \end{array}\right], \tag{22}$$

while the degree matrix D_M of H is given as

$$D_M = \left[\begin{array}{cc} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{array}\right]. \tag{23}$$

On the other hand, the ordinary degree matrix defined by (2) for $\tilde{A}$ is reduced to

$$\tilde{D} = \left[\begin{array}{cccc} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right], \tag{24}$$

which is equivalent to D_K whose components are replaced by 2×2 real matrices as (18), which supports the validity of D_K for H, as it reflects the one-to-one correspondence of A and $\tilde{A}$ [9]. Hereafter, we call $\tilde{L} := \tilde{D} - \tilde{A}$ a duplicated- nodes graph Laplacian.

IV. EQUIVALENCE OF GRAPH SIGNAL PROCESSING BY L_K AND $\tilde{L}$

In this section, we discuss the relationship between the GFTs based on L_K and $\tilde{L}$ and the relationship between graph signal processing using these GFTs, which has not been discussed in [9].

A unit real eigenvector of a real symmetric matrix corresponding to a simple eigenvalue has degrees of freedom in its direction, that is, if $\boldsymbol{u}$ is a real unit eigenvector, then $-\boldsymbol{u}$ is also a real unit eigenvector. Similarly, a unit complex eigenvector of a complex Hermitian matrix corresponding to a simple eigenvalue can have an arbitrary sign $\{\exp(i\theta) \mid \theta \in [-\pi, \pi]\}$, that is, if $\boldsymbol{u}$ is a unit eigenvector, then $\exp(\theta)\boldsymbol{u}$ is also a unit eigenvector for any $\theta \in [-\pi, \pi]$.

Let

$$L_K = U\Lambda U^* \tag{25}$$

be the eigenvalue decomposition of a Hermitian graph Laplacian L_K. By taking into account the degrees of freedom for signs of eigenvectors, a matrix US is also a unitary matrix consisting of the eigenvectors of L_K, where $S \in \mathbf{C}^N$ is a diagonal matrix whose i-th diagonal element is the sign $\exp(\theta_i)$, $(\theta_i \in [-\pi, \pi])$. Therefore, (25) can also be written as

$$L_K = (US)\Lambda(US)^*. \tag{26}$$

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[4] It should be noted that even if normalization is performed, the following discussion will not fall apart. We omit normalization in order for simplicity of descriptions.

Here, we have to clarify whether S affects the result of signal processing or not.

Proposition 1: The result of signal processing (17) is invariant for any diagonal matrix S whose diagonal elements are unit complex numbers if $\Phi(\Lambda) \in \mathcal{D}$.

Proof If $\Phi(\Lambda) \in \mathcal{D}$, $S\Phi(\Lambda)S^* = \Phi(\Lambda)SS^* = \Phi(\Lambda)$ since S is also diagonal. Therefore, we have

$$\begin{aligned}\hat{\boldsymbol{f}} &= (US)\Phi(\Lambda)(US)^*\boldsymbol{f} = U(S\Phi(\Lambda)S^*)U^*\boldsymbol{f} \\ &= U\Phi(\Lambda)U^*\boldsymbol{f},\end{aligned}$$

which concludes the proof. □

According to Proposition 1, it is concluded that the sign of the eigenvector of L_K does not affect the result of signal processing as long as $\Phi(\Lambda) \in \mathcal{D}$. However, it should be noted that when $\Phi(\Lambda) \notin \mathcal{D}$, which means that a signal processing operation is across two or more frequencies, the sign of the eigenvector may affect the result of signal processing.

Let $\tilde{L}$, $\tilde{U}$, $\tilde{S}$ and $\tilde{\Lambda}$ be $2N \times 2N$ real matrices obtained by replacing each element of L_K, U, S and Λ in (26) with the corresponding 2×2 matrices (18). Then,

$$\tilde{L} = \tilde{U}\tilde{S}\tilde{\Lambda}\tilde{S}'\tilde{U}' \tag{27}$$

trivially holds due to the correspondence of operations for complex numbers and those for 2×2 matrices (18). In addition,

$$\tilde{U}\tilde{S}\tilde{S}'\tilde{U}' = \tilde{S}'\tilde{U}'\tilde{U}\tilde{S} = I_{2N}$$

holds since $USS^*U^* = U^*S^*SU = I_N$, and $\tilde{\Lambda} \in \mathbf{R}^{2N\times 2N}$ is a real diagonal matrix since eigenvalues λ of a Hermitian matrix are real and its corresponding 2×2 matrix corresponds to λI_2. Therefore, it is concluded that (27) gives the eigenvalue decomposition of $\tilde{L}$.

Let $\tilde{F} := \boldsymbol{f} \otimes I_2 \in \mathbf{R}^{2N\times 2}$, with $\otimes$ denoting the Kronecker product [12] of matrices (or vectors), be a signal vector $\boldsymbol{f}$ whose i-th element is replaced by the corresponding 2×2 matrix $f_i I_2$. Then, the GFT and its inverse based on the eigenstructure of the duplicated-nodes graph Laplacian $\tilde{L}$ are written as

$$\tilde{F}_G := \tilde{U}'\tilde{F} \tag{28}$$

$$\tilde{F} := \tilde{U}\tilde{F}_G, \tag{29}$$

and they are essentially identical to (15) and (16) due to the above argument. Note that $\tilde{F}$ can be replaced by $\tilde{\boldsymbol{f}} := \boldsymbol{f} \otimes \boldsymbol{e}_1$ (or $\tilde{\boldsymbol{f}} := \boldsymbol{f} \otimes \boldsymbol{e}_2$), where $\boldsymbol{e}_i$ stands for the 2-dimensional unit vector whose i-th element is unity, and then $\tilde{F}_G$ can also be replaced by $\tilde{\boldsymbol{f}}_G := \boldsymbol{f}_G \otimes \boldsymbol{e}_1$ (or $\tilde{\boldsymbol{f}}_G := \boldsymbol{f}_G \otimes \boldsymbol{e}_2$), since $\tilde{F}$ and $\tilde{F}_G$ have redundant information as shown in (18). Therefore, graph signal processing based on the eigenstructure of $\tilde{L}$ can be represented as

$$\hat{\tilde{\boldsymbol{f}}} = \tilde{U}\tilde{S}\tilde{\Phi}(\tilde{\Lambda})\tilde{S}'\tilde{U}'\tilde{\boldsymbol{f}}, \tag{30}$$

where $\tilde{\Phi}$ is a mapping defined on $\tilde{\mathcal{D}} := \{\Lambda \otimes I_2 \mid \Lambda \in \mathcal{D}\}$ that produces a $2N \times 2N$ matrix, which represents a signal processing procedure in the graph Fourier domain of $\tilde{L}$. Note that $\tilde{S}$ in (30) can be omitted for the same reason as the cases for signal processing based on a Hermitian graph Laplacian L_K.

Here, we have an issue that should be discussed, that is, new degrees of freedom of $\tilde{U}$, coming from duplicated eigenvalues of $\tilde{L}$.

Since all the eigenvalues of $\tilde{L}$ are duplicated ones with multiplicity 2, the dimension of the corresponding eigenspace is 2, which implies that we can choose arbitrary linearly independent two vectors in the eigenspace as its basis. Therefore, (30) can be generalized as

$$\hat{\tilde{\boldsymbol{f}}} = \tilde{U}\tilde{T}\tilde{\Phi}(\tilde{\Lambda})\tilde{T}^{-1}\tilde{U}'\tilde{\boldsymbol{f}}, \tag{31}$$

where T denotes an arbitrary $2N \times 2N$ block diagonal matrix whose diagonal blocks are arbitrary 2×2 non-singular matrices. Note that $\tilde{U}\tilde{T} \in \mathbf{R}^{2N\times 2N}$ may be no longer an orthogonal matrix. Thus, it should be clarified whether T affects the result of signal processing based on the eigen structure of $\tilde{L}$ or not.

Proposition 2: The result of signal processing (31) is invariant for any block diagonal matrix T whose 2×2 diagonal blocks are non-singular if $\tilde{\Phi}(\tilde{\Lambda}) \in \tilde{\mathcal{D}}$.

Proof Let $T_i \in \mathbf{R}^{2\times 2}$, $(i \in \{1, \ldots, N\})$ be an arbitrary non-singular matrix, and let

$$\tilde{T} := \begin{bmatrix} T_1 & O_2 & \cdots & O_2 \\ O_2 & T_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O_2 \\ O_2 & \cdots & O_2 & T_N \end{bmatrix}, \tag{32}$$

where O_2 stands for the 2×2 zero matrix. Then, it is trivial that

$$\tilde{T}^{-1} := \begin{bmatrix} T_1^{-1} & O_2 & \cdots & O_2 \\ O_2 & T_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O_2 \\ O_2 & \cdots & O_2 & T_N^{-1} \end{bmatrix}. \tag{33}$$

Therefore, if $\tilde{\Phi}(\tilde{\Lambda}) \in \tilde{\mathcal{D}}$,

$$\tilde{T}\tilde{\Phi}(\Lambda)\tilde{T}^{-1} = \tilde{\Phi}(\Lambda)\tilde{T}\tilde{T}^{-1} = \tilde{\Phi}(\Lambda), \tag{34}$$

since $\tilde{\Phi}(\tilde{\Lambda}) \in \tilde{\mathcal{D}}$. Therefore,

$$\begin{aligned}\hat{\tilde{\boldsymbol{f}}} &= \tilde{U}\tilde{T}\tilde{\Phi}(\tilde{\Lambda})\tilde{T}^{-1}\tilde{U}'\tilde{\boldsymbol{f}} \\ &= \tilde{U}\tilde{\Phi}(\tilde{\Lambda})\tilde{U}'\tilde{\boldsymbol{f}},\end{aligned}$$

which concludes the proof. □

According to Proposition 2, it is concluded that the degrees of freedom, coming from duplicated eigenvalues, do not affect signal processing based on the duplicated-nodes graph Laplacian $\tilde{L}$. However, it should be noted that when $\tilde{\Phi}(\tilde{\Lambda}) \notin \tilde{\mathcal{D}}$, T may affect the result of signal processing as is the case with the signal processing based on the eigenstructure of a Hermitian graph Laplacian L_K.

From the above discussions, we conclude that a lot of knowledge on graph signal processing for undirected graphs can be applied to that for directed graphs if we adopt the degree matrix D_K for a Hermitian graph Laplacian and consider the corresponding duplicated-nodes graph Laplacian, as long as $\tilde{\Phi}(\tilde{\Lambda})$ is diagonal.

V. CONCLUSIONS

In this paper, we analyzed the relationship between the eigenstructures of a Hermitian graph Laplacian and its corresponding duplicated-nodes graph Laplacian, and demonstrated that signal processing based on these two Laplacians is essentially equivalent. A remaining issue to be addressed is the interpretation of the results of signal processing based on these two Laplacians.

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REFERENCES

[1] A. Ortega, *Introduction to Graph Signal Processing*. Cambridge University Press, 2022.

[2] Y. Tanaka, “Recent Advances in Signal Processing on Graphs,” *IEICE Fundamentals Review*, vol. 8, no. 1, pp. 15–29, 2014.

[3] A. Sandryhaila and J. M. F. Moura, “Discrete Signal Processing on Graphs: Frequency Analysis,” *IEEE Transactions on Signal Processing*, vol. 62, no. 12, pp. 3042–3054, 2014.

[4] R. Singh, A. Chakraborty, and B. S. Manoj, “Graph Fourier Transform Based on Directed Laplacian,” in *2016 International Conference on Signal Processing and Communications (SPCOM)*, 2016, pp. 1–5.

[5] S. Sardellitti, S. Barbarossa, and P. D. Lorenzo, “On the Graph Fourier Transform for Directed Graphs,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 11, no. 6, pp. 796–811, 2017.

[6] C. Cheng, Y. Chen, Y. J. Lee, and Q. Sun, “SVD-Based Graph Fourier Transforms on Directed Product Graphs,” *IEEE Transactions on Signal and Information Processing over Networks*, vol. 9, pp. 531–541, 2023.

[7] Y. Sugie and A. Tanaka, “Graph Fourier Transform on Directed Graphs Based on Hermitian Graph Laplacian,” in *Proc. of SIP Symposium 2019*, 2019, pp. 131–132.

[8] M. Muramatsu, H. Kitamura, and Y. Tanaka, “Generalization of Graph Fourier Transform for Directed Graph Signal Processing,” in *Proc. of SIP Symposium 2022*, 2022, pp. 418–423.

[9] S. Kamada and A. Tanaka, “Normalized Hermitian Graph Laplacian and Its Properties,” in *Proc. of SIP Symposium 2023*, 2023, 6 pages.

[10] S. Furutani, T. Shibahara, M. Akiyama, K. Hato, and M. Aida, “Graph Signal Processing for Directed Graphs Based on the Hermitian Laplacian,” in *Machine Learning and Knowledge Discovery in Databases: European Conference, ECML PKDD 2019, Proceedings, Part I*, Springer, 2020, pp. 447–463.

[11] H. Kitamura, H. Yasuda, Y. Tanaka, and M. Muramatsu, “Realization of Digraph Filters via Augmented GFT,” in *Proc. of International Conference on Image Processing (ICIP2023*, 2023, pp. 2885–2889.

[12] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Second. John Wiley, 1999.