Phase Retrieval – A Deconvolution Perspective

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***Abstract*—Phase retrieval finds applications in various optical imaging modalities such as X-ray crystallography, holography, frequency-domain optical-coherence tomography, etc.. The sensors used in optical imaging can measure only the magnitudes of incoming wavefronts and the phase information is not measured directly. This necessitates developing appropriate phase retrieval algorithms to reconstruct the object as phase contains most of the structural information. The phase retrieval problem naturally arises in the Fourier imaging context, where the measurement is the Fourier magnitude/intensity spectrum. Reconstruction from the Fourier intensity results in the autocorrelation and not the signal. We therefore address the equivalent problem of signal retrieval from the autocorrelation. Since the signal autocorrelation can be expressed as a convolution of the signal with its flipped version, we propose to solve the phase retrieval problem within a deconvolution framework. We consider a non-convex cost in two vector variables, the signal and its flipped version. An alternating minimization (Alt. Min.) strategy is employed to arrive at an optimal estimate of the signal, given the autocorrelation. Due to non-convexity of the cost function, the accuracy of the estimation is critically dependent on the initialization. We establish that the Alt. Min. iterates ensure that the cost is nonincreasing. For the specific case of causal, delta-dominant signals, the proposed framework results in exact reconstruction with an all zero-phase initialization. We shall also consider the effect of random initialization on the estimation accuracy.**

I. Introduction

Phase retrieval is the problem of reconstructing a signal from measurements of the Fourier magnitude spectrum alone. Phase retrieval is encountered in numerous engineering and scientific applications such as crystallography [1], holography [2], electron microscopy [3], frequency-domain optical-coherence tomography (FDOCT) [4], etc., where only the intensities of the complex-valued signals are measured and the phase information is lost. In most optical imaging applications, the phase of the Fourier transform carries critical structural information about the object, thus making phase retrieval inevitable.

Since an infinite number of signals have the same magnitude spectrum, the problem of phase retrieval is ill-posed. To determine the signal that led to the magnitude measurements, phase retrieval techniques make use of prior information about the signal, such as its support, casuality, or sparsity, or rely on over-sampled Fourier measurements. Before formally stating the problem addressed in this paper, we give a brief overview of the relevant literature on phase retrieval.

A. Related Work

The challenge of developing efficient, guaranteed methods for phase retrieval has attracted substantial interest over the past decade [5]. The initial techniques are due to Gerchberg and Saxton [6] and Fienup [7], and are based on the idea of error-reduction, where the estimate is alternately projected between the measurement domain and the signal domain applying appropriate constraints in respective domains. More insights have been developed along this line of research by considering convexity [8] and applying proximal optimization strategies [9]. Since phase retrieval is an ill-posed problem, the need to find a unique solution has promoted extensive study of the class of minimum-phase signals. The well-known result of Hilbert transform relation between the log-magnitude and phase spectra associated with the minimum-phase sequences [10] forms the foundation of the retrieval algorithms developed in [11], [12]. Both iterative [13] and non-iterative [14]–[16] techniques have been developed, which result in exact phase retrieval for minimum-phase signal models. Deviating from the notion of minimum phase, Shenoy et al. [17] introduced a new class of signals, called causal, delta-dominant (CDD) sequences, which allow for exact phase retrieval.

Recently, phase retrieval has been studied extensively by the compressed sensing (CS) community. Notable CS based algorithms are the compressive phase retrieval (CPR) algorithm by Moravec et al. [18], and a greedy sparse phase retrieval (GESPAR) algorithm by Shechtman et al. [19]. Most sparsity-driven phase retrieval algorithms reconstruct the signal up to a global phase factor [19], [20]. To ensure uniqueness, Ohlsson and Eldar [21] established the conditions of sufficiency on the measurements. Considering undersampled measurements, Weller et al. [22] developed a multi-layered transform-domain approach based on iterative optimization, which is used in the reconstruction of sparse images. Lu and Vetterli [23] addressed the problem of sparse spectral factorization, which is a different perspective of the phase retrieval problem, and perhaps the most closely related one to the framework considered in this paper.

Other approaches to phase retrieval employ convex relaxation, where phase retrieval is reformulated as a rank-minimization problem followed by a relaxation to nuclear-norm or trace minimization [24]–[26]. Semidefinite-programming-based techniques have been popular within this framework [27], [28]. Non-convex formulations for phase retrieval based on suitable initialization and gradient update have also been developed [29], [30]. Most non-convex techniques that do not resort to convex relaxations come under the umbrella of alternating minimization algorithms. Netrapalli et al. [31] studied the alternating projection schemes and provided convergence guarantees under certain types of initialization. Mukherjee and Seelamantula [32] addressed the problem of sparse phase retrieval, which employs alternating minimization based on modified Fienup iterations. Yang et al. [33] proposed a method of alternating directions, which considers CPR problem in an image restoration framework. More recently, Mukherjee and Seelamantula [34] posed the problem of phase retrieval in a variable splitting framework.

B. This Paper

The autocorrelation of a signal and its Fourier intensity (power spectrum) are related by means of the Fourier transform. The autocorrelation is inherently phase-blind, which encourages one to recast the problem of phase retrieval in spatial domain, deviating from the often employed Fourier domain formulation. This idea has been studied popularly as spectral factorization, and relies on the fundamental property that the autocorrelation of a signal is the convolution of the signal with its flipped version. In Section II, we develop a novel deconvolution framework for the problem of phase retrieval. The deconvolution considered here is necessarily a blind-deconvolution and is therefore a non-convex problem. The non-convexity is circumvented by considering two simpler convex problems (Section III), which leads to an Alt. Min. strategy. The accuracy of the estimate is critically dependent on the initialization as the cost function has multiple local minima. In particular, we investigate both all zero-phase initialization as well as random initialization. Further, we establish the non-increasing property of the cost function (Section IV). The class of causal, delta-dominant (CDD) signals admits a unique solution up to a global phase factor. The proposed approach exactly recovers the ground-truth signal in such scenarios with an all-zero-phase initialization. As shown in this paper, certain types of random initialization also result in accurate recovery for CDD sequences.

II. Problem Formulation

Signals are uniquely characterized by their Fourier magnitude and phase spectra. Fourier intensity measurements alone are incomplete to reconstruct the underlying signal because of the missing phase information. Combining different phase spectra with the measurements of the magnitude spectrum results in different signals. Hence, oversampling becomes inevitable for successful phase retrieval. In this paper, we consider oversampled Fourier measurements. The problem is about recovering a signal $\mathbf{x} \in \mathbb{R}^N$ from the measurements $|\mathbf{F}_M\mathbf{x}|^2$, where $\mathbf{F}_M$ is constructed using the first N columns of the M-point DFT (discrete Fourier transform) matrix, $M \geq 2N-1$ and $|\cdot|^2$ is computed elementwise. Let $\mathbf{r} \in \mathbb{R}^{2N-1}$ denote the autocorrelation sequence of $\mathbf{x}$, which forms a Fourier pair with $|\mathbf{F}_M\mathbf{x}|^2$. The autocorrelation $\mathbf{r}$ is defined as

$$\begin{aligned}\mathbf{r} &= \mathbf{x} * \text{flip}(\mathbf{x}),\\ &= \mathbf{x} * \mathbf{P}\mathbf{x},\end{aligned} \tag{1}$$

where $*$ denotes the linear convolution operator and $\mathbf{P} = (p_{ij}) \in \mathbb{R}^{N\times N}$ with

$$p_{ij} = \begin{cases} 1, & i+j = N+1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

denotes the flip operator. We rewrite (1) as

$$\mathbf{r} = \mathbf{x} * \mathbf{y} = \mathbf{X}\mathbf{y} = \mathbf{Y}\mathbf{x}, \tag{3}$$

where $\mathbf{y} = \mathbf{P}\mathbf{x}$ and $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{2N-1\times N}$ are the linear convolution matrices constructed from $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^N$, respectively. Estimating the signal $\mathbf{x}$ using the measurement model in (3) is a blind deconvolution problem, which is inherently ill-posed as there exist infinitely many combinations of $\mathbf{x}$ and $\mathbf{y}$ that give rise to the same $\mathbf{r}$. Taking into account $\mathbf{y} = \mathbf{P}\mathbf{x}$ would constrain the solution space, but there still exist an infinite number of solutions. Consider the following deconvolution formulation of the phase retrieval problem:

$$\begin{aligned} &\text{Find} && \mathbf{x}, \\ &\text{such that} && \mathbf{r} = \mathbf{x} * \mathbf{y}, \\ &\text{where} && \mathbf{y} = \mathbf{P}\mathbf{x}. \end{aligned} \tag{4}$$

Let us cast (4) into an optimization framework using the following least-squares cost function

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^N}{\text{minimize}} && \left\|\mathbf{r} - \mathbf{x} * \mathbf{y}\right\|^2, \\ &\text{subject to} && \mathbf{y} = \mathbf{P}\mathbf{x}, \end{aligned} \tag{5}$$

which can further be expressed in an unconstrained form as follows:

$$\underset{\mathbf{x},\mathbf{y}\in\mathbb{R}^N}{\text{minimize}} \quad \underbrace{\left\|\mathbf{r} - \mathbf{x} * \mathbf{y}\right\|^2 + \left\|\mathbf{P}\mathbf{x} - \mathbf{y}\right\|^2}_{F(\mathbf{x},\mathbf{y})}. \tag{6}$$

The blind deconvolution problem stated in (6) is non-convex and hence, one cannot resort to standard convex optimization methods to solve it.

III. Alt. Min. Optimization

We adopt an alternating minimization strategy to solve the optimization problem in (6). In the first step, we fix $\mathbf{x}$ and optimize $F(\mathbf{x}, \mathbf{y})$ over $\mathbf{y}$ (flipped signal optimization or *f-step*). In the next step, $F(\mathbf{x}, \mathbf{y})$ is updated with the estimate of $\mathbf{y}$ and then optimized over $\mathbf{x}$ (actual signal optimization or *a-step*).

Let $\mathbf{x}^{(k)}$ denote the estimate obtained at the end of k^{th} iteration. In the *f-step*, $F\left(\mathbf{x}^{(k)}, \mathbf{y}\right)$ is optimized with respect to $\mathbf{y}$ to obtain $\mathbf{y}^{(k+1)}$:

$$\mathbf{y}^{(k+1)} = \arg\min_{\mathbf{y}} \quad \underbrace{\left\|\mathbf{r} - \mathbf{X}^{(k)}\mathbf{y}\right\|^2 + \left\|\mathbf{P}\mathbf{x}^{(k)} - \mathbf{y}\right\|^2}_{F\left(\mathbf{x}^{(k)},\mathbf{y}\right)}, \tag{7}$$

Algorithm 1 Alt. Min. Deconvolution for Phase Retrieval

- **Input**: Autocorrelation measurement $\mathbf{r} \in \mathbb{R}^{2N-1}$.
- **Initialization**: $k = 0$, $\mathbf{x}^{(0)} \in \mathbb{R}^N$, $\mathbf{X}^{(0)} =$ Conv. Matrix $\left(\mathbf{x}^{(0)}\right)$, k_{max}.
- **While** $(k \leq k_{\text{max}})$ **do**:
  1) ***f-step***: $\mathbf{y}^{(k+1)} = \left(\mathbf{X}^{(k)\mathrm{T}}\mathbf{X}^{(k)} + \mathbf{I}\right)^{-1} \left(\mathbf{X}^{(k)\mathrm{T}}\mathbf{r} + \mathbf{P}\mathbf{x}^{(k)}\right)$.
  2) Construct $\mathbf{Y}^{(k+1)} =$ Conv. Matrix $\left(\mathbf{y}^{(k+1)}\right)$.
  3) ***a-step***: $\mathbf{x}^{(k+1)} = \left(\mathbf{Y}^{(k+1)\mathrm{T}}\mathbf{Y}^{(k+1)} + \mathbf{I}\right)^{-1} \left(\mathbf{Y}^{(k+1)\mathrm{T}}\mathbf{r} + \mathbf{P}\mathbf{y}^{(k+1)}\right)$.
  4) Update $\mathbf{X}^{(k+1)} =$ Conv. Matrix $\left(\mathbf{x}^{(k+1)}\right)$.
  5) $k \leftarrow k + 1$.

  end while
- **Output**: $\mathbf{x}$.

where $\mathbf{X}^{(k)} \in \mathbb{R}^{2N-1\times N}$ is a linear convolution matrix constructed from $\mathbf{x}^{(k)}$. The cost function $F\left(\mathbf{x}^{(k)}, \mathbf{y}\right)$ is differentiable with respect to $\mathbf{y}$ and has a stationary point $\bar{\mathbf{y}}$, corresponding to which

$$\mathbf{X}^{(k)\mathrm{T}}\left(\mathbf{r} - \mathbf{X}^{(k)}\bar{\mathbf{y}}\right) + \mathbf{P}\mathbf{x}^{(k)} - \bar{\mathbf{y}} = 0. \quad (8)$$

Therefore,

$$\begin{aligned}\bar{\mathbf{y}} &= \left(\mathbf{X}^{(k)\mathrm{T}}\mathbf{X}^{(k)} + \mathbf{I}\right)^{-1} \left(\mathbf{X}^{(k)\mathrm{T}}\mathbf{r} + \mathbf{P}\mathbf{x}^{(k)}\right), \\ &= \mathbf{y}^{(k+1)},\end{aligned} \quad (9)$$

which is used to update $\mathbf{x}^{(k+1)}$ in the *a-step* as follows:

$$\begin{aligned}\mathbf{x}^{(k+1)} = &\quad \arg\min_{\mathbf{x}} \quad F\left(\mathbf{x}, \mathbf{y}^{(k+1)}\right), \\ = &\quad \arg\min_{\mathbf{x}} \quad \left\|\mathbf{r} - \mathbf{Y}^{(k+1)}\mathbf{x}\right\|^2 + \left\|\mathbf{P}\mathbf{x} - \mathbf{y}^{(k+1)}\right\|^2,\end{aligned} \quad (10)$$

where $\mathbf{Y}^{(k+1)} \in \mathbb{R}^{2N-1\times N}$ is a linear convolution matrix constructed from $\mathbf{y}^{(k+1)}$. The cost function $F\left(\mathbf{x}, \mathbf{y}^{(k+1)}\right)$ is differentiable with respect to $\mathbf{x}$ and has a stationary point $\bar{\mathbf{x}}$, corresponding to which

$$\mathbf{Y}^{(k+1)\mathrm{T}}\left(\mathbf{r} - \mathbf{Y}^{(k+1)}\bar{\mathbf{x}}\right) - \mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y}^{(k+1)}\right) = 0. \quad (11)$$

Since $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{I}$ and $\mathbf{P}^{\mathrm{T}} = \mathbf{P}$, we have

$$\begin{aligned}\bar{\mathbf{x}} &= \left(\mathbf{Y}^{(k+1)\mathrm{T}}\mathbf{Y}^{(k+1)} + \mathbf{I}\right)^{-1} \left(\mathbf{Y}^{(k+1)\mathrm{T}}\mathbf{r} + \mathbf{P}\mathbf{y}^{(k+1)}\right), \\ &= \mathbf{x}^{(k+1)}.\end{aligned} \quad (12)$$

The alternating minimization scheme is summarized in Algorithm 1, which breaks a non-convex problem into two simpler convex sub-problems.

A. Descent Property of the Cost Function

We next establish that, after every update of $\mathbf{x}^{(k)}$ and $\mathbf{y}^{(k)}$, the cost $F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\right)$ is a non-increasing function, which ensures that, in practice, the iterations converge to a reasonable solution. We first consider the behavior of the cost function F for a fixed $\mathbf{x}^{(k)}$, but with the flipped signal estimated (Lemma 1), then with a fixed $\mathbf{y}^{(k)}$ and the actual signal estimated (Lemma 2). Finally, we combine the two results to get the desired descent property (Lemma 3).

Lemma 1. *Let $\mathbf{y}^{(k+1)}$ be the minimizer of $F\left(\mathbf{x}^{(k)}, \mathbf{y}\right)$ after $(k+1)^{st}$ iteration, for a fixed $\mathbf{x}^{(k)}$. Then, F satisfies the descent property*

$$F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k+1)}\right) \leq F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\right). \quad (13)$$

Proof: The Hessian of the cost function $F\left(\mathbf{x}^{(k)}, \mathbf{y}\right) = \left\|\mathbf{r} - \mathbf{X}^{(k)}\mathbf{y}\right\|^2 + \left\|\mathbf{P}\mathbf{x}^{(k)} - \mathbf{y}\right\|^2$ is $2\left(\mathbf{X}^{(k)\mathrm{T}}\mathbf{X}^{(k)} + \mathbf{I}\right)$, which is a positive-definite matrix. Consequently $F\left(\mathbf{x}^{(k)}, \mathbf{y}\right)$ is strictly convex and hence, $F\left(\mathbf{x}^{(k)}, \mathbf{y}\right)$ has a unique minimizer, which we denote as $\mathbf{y}^{(k+1)}$. Thus, $F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k+1)}\right) \leq F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\right)$. ■

Lemma 2. *Let $\mathbf{x}^{(k+1)}$ be the minimizer of $F\left(\mathbf{x}, \mathbf{y}^{(k+1)}\right)$ after $(k+1)^{st}$ iteration, for a fixed $\mathbf{y}^{(k+1)}$. Then, F satisfies the descent property*

$$F\left(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}\right) \leq F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k+1)}\right). \quad (14)$$

Proof: The Hessian of the cost function $F\left(\mathbf{x}, \mathbf{y}^{(k+1)}\right) = \left\|\mathbf{r} - \mathbf{Y}^{(k+1)}\mathbf{x}\right\|^2 + \left\|\mathbf{P}\mathbf{x} - \mathbf{y}^{(k+1)}\right\|^2$ is $2\left(\mathbf{Y}^{(k+1)\mathrm{T}}\mathbf{Y}^{(k+1)} + \mathbf{I}\right)$, which is a positive-definite matrix. Consequently $F\left(\mathbf{x}, \mathbf{y}^{(k+1)}\right)$ is strictly convex and hence, $F\left(\mathbf{x}, \mathbf{y}^{(k+1)}\right)$ has a unique minimizer, which we denote as $\mathbf{x}^{(k+1)}$. Thus, $F\left(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}\right) \leq F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k+1)}\right)$. ■

Combining Lemmas 1 and 2 gives the following result pertaining to the descent of the cost F after updating both the flipped and the actual signals.

Lemma 3. *Suppose $\mathbf{y}^{(k+1)}$ and $\mathbf{x}^{(k+1)}$ are the minimizers in* (7) *and* (10)*, respectively. After $(k+1)^{st}$ iteration of Algorithm 1,*

$$F\left(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}\right) \leq F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\right). \quad (15)$$

While the cost $F\left(\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\right)$ is guaranteed not to increase with iterations and the algorithm is relatively simple, the non-convexity of (6) in the variables $\mathbf{x}$ and $\mathbf{y}$, and the presence of local minima is a concern.

Algorithm 2 Estimation of Zero-Phase Initialization [35]

- **Input**: Autocorrelation measurement $\mathbf{r} = [r_{1-N}, \ldots, r_{-1}, r_0, r_1, \ldots, r_{N-1}] \in \mathbb{R}^{2N-1}$.
- **Method**:
  1) Obtain the one-sided autocorrelation $\bar{\mathbf{r}} = [r_0, r_1, \ldots, r_{N-1}] \in \mathbb{R}^N$.
  2) Construct the diagonal matrix $\tilde{\mathbf{I}}_{N\times N} = \text{diag}\{[1\ 2\ 2 \ldots 2]\}$ and compute the N-point DFT of $\tilde{\mathbf{I}}\bar{\mathbf{r}}$: $\mathbf{F}\tilde{\mathbf{I}}\bar{\mathbf{r}}$, where $\mathbf{F}$ denotes the N-point DFT matrix.
  3) Compute the zero phase sequence: $\boldsymbol{\phi} = [0, 0, \ldots, 0]$.
  4) Reconstruct $\mathbf{x}^{(0)}$: $\mathbf{F}^{-1}\{\mathcal{R}\{\mathbf{F}\tilde{\mathbf{I}}\bar{\mathbf{r}}\}^{\frac{1}{2}} e^{\mathrm{j}\boldsymbol{\phi}}\}$, where $\mathcal{R}\{.\}$ takes the real part of its argument.
- **Output**: $\mathbf{x}^{(0)}$.

B. Initialization

Due to non-convexity of the cost function, convergence of the algorithm to the local minima is largely governed by the initialization $\mathbf{x}^{(0)}$. Employing random initialization for the initial estimate of the actual signal leaves us uncertain about the final estimate $\mathbf{x}^*$, which could either be a local minimum or one of the global minima (note that the cost function in general has several global minima). For signals such as minimum-phase signals and causal, delta-dominant (CDD) signals that are retrievable uniquely up to a global phase factor from the magnitude spectrum, this approach would give the unique solution.

For initialization, we determine $\mathbf{x}^{(0)}$ that corresponds to the inverse Fourier transform of the measurements of the magnitude (not intensity) spectrum (with zero-phase component), which is obtained from the autocorrelation measurements [35]. We call such an estimate as zero-phase initialization and the estimation procedure is summarized in Algorithm 2. In the speech processing context, this is also referred to as *square-root autocorrelation*.

IV. Experiments and Results

We empirically demonstrate the usefulness of zero-phase initialization as one of the ways to avoid bad initializations that lead to local minima. Banking on the descent property of the cost with zero-phase initialization, we analyze exactness of phase retrieval for CDD signals. The CDD class of signals has exact phase retrieval properties similar to minimum-phase signals [17]. Finally, we explore both adversity and advantage of deviating from the zero-phase initialization.

A. Effect of Zero-Phase Initialization

We start with the autocorrelation of a randomly generated 256-length sequence. An all-zero phase is used to begin the iterations of the Alt. Min. scheme. The value of the objective as a function of iterations, averaged over 100 independent trials is shown in Fig 1. Empirically, the cost function was found to decrease monotonically and attain a value close to zero. Thus, we empirically show that zero-phase initialization succeeds in recovering the sequence $\mathbf{x}$, which is one of the global minima of the deconvolution problem (6).

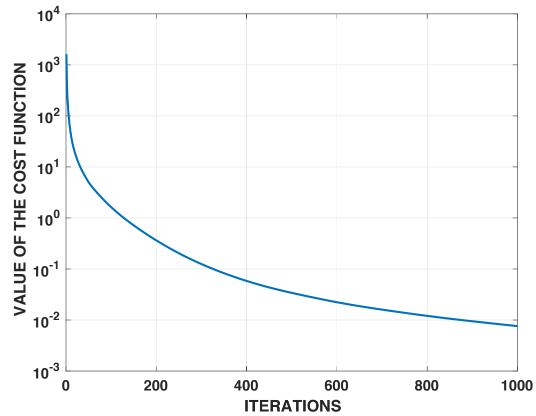

Fig. 1: Convergence to a global minimum using zero-phase initialization. The profile has been obtained by averaging over 100 independent Monte Carlo trials.

To assess the performance of the algorithm on synthesized data, we define the normalized mean-squared-error (NMSE) metric. Let $\hat{\mathbf{x}}$ be an estimate of the ground-truth signal $\mathbf{x}$. The NMSE of the estimate is defined as

$$\text{NMSE} = 20\ \log_{10}\left(\frac{\|\mathbf{x}-\hat{\mathbf{x}}\|}{\|\mathbf{x}\|}\right) \text{dB}. \tag{16}$$

B. Reconstruction of CDD Sequences

Definition 1 (Causal, delta-dominant sequence [17])**.** A sequence $\mathbf{z} = \{z_n\}_{n\in\mathbb{Z}}$ is said to be causal delta-dominant (CDD) if $z_n = 0$, for $n < 0$ and $z_0 > \sum_{n>0} |z_n|$.

A CDD sequence need not have a rational transfer function [17], unlike a minimum-phase sequence. However, there exist sequences that are both CDD and minimum phase. Specifically, a finite-length CDD sequence is also a minimum-phase sequence [17].

We now use zero-phase initialization to analyze the alternating minimization procedure, when the autocorrelation measurements of a CDD sequence are available. A 256-length CDD sequence is randomly generated, which is used to obtain the measurements. The experiment is repeated for 100 independent trials. Figs. 2(c) and (d) confirm that the cost reduces to a small value. The NMSE plots are shown in Figs. 2(a) and (b), which shows that the algorithm achieves a reconstruction error of -57 dB. In Fig. 3, we demonstrate signal reconstruction taking the specific example of a CDD sequence shown in Fig. 3(a). The reconstructed sequence is shown in Fig. 3(b). The proposed reconstruction is seen to have an MSE of the order of magnitude 10^{-30}. This shows that CDD sequences are exactly reconstructed by the proposed algorithm with zero-phase initialization.

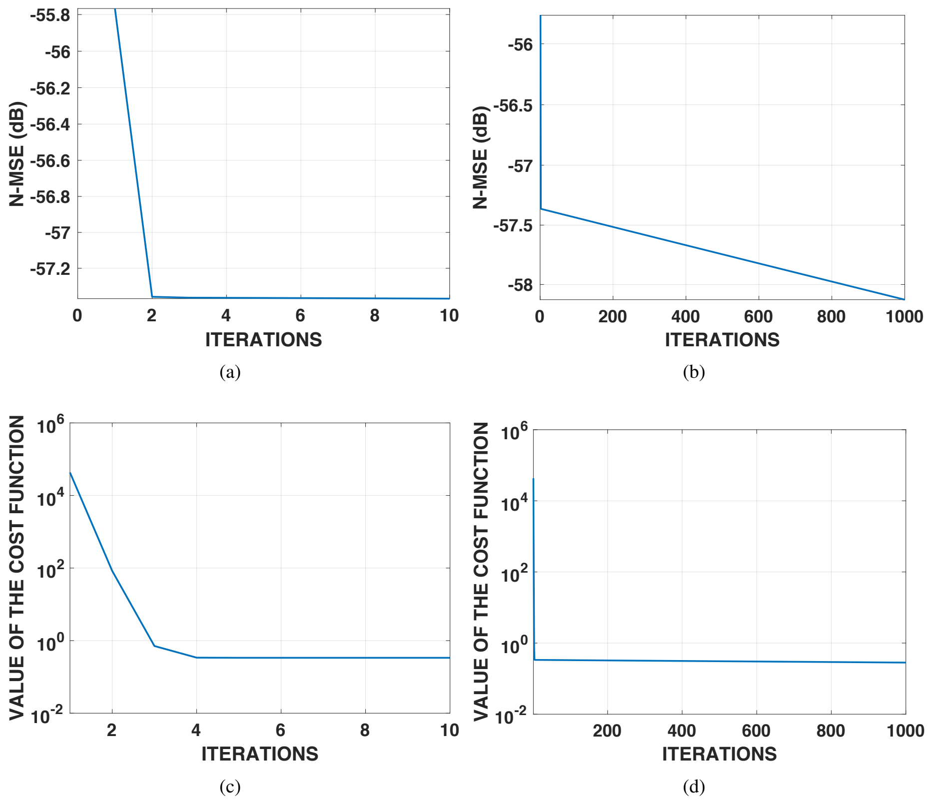

Fig. 2: Phase retrieval of a CDD sequence with zero-phase initialization. The values are averaged over 100 independent trials. (a) NMSE for 10 iterations; (b) NMSE for 1000 iterations; (c) Cost function profile over 10 iterations; and (d) Cost function profile over 1000 iterations.

C. Deviating From the Zero-Phase Initialization

We investigate the significance of zero-phase initialization, where we address two questions: i) Is there any other initialization that guarantees exact reconstruction of a CDD sequence? and ii) Does zero-phase initialization ensure the best convergence? Instead of combining zero-phase in Algorithm 2, we consider a random phase sequence drawn from a uniform distribution in the interval $[-\frac{\pi}{Q}, \frac{\pi}{Q}]$, $Q \geq 1$ as the initial estimate $\mathbf{x}^{(0)}$. Fig. 4(a) shows five such distributions obtained with different values of Q. We start with the measurements corresponding to a 256-length CDD sequence. In Fig 4(b), we have recorded the reconstruction performance, averaged over 100 independent trials for different Qs, which shows that for $Q \geq 4$, the proposed recovery method attains a reconstruction error better than -40 dB. Also, superior reconstruction is obtained as Q increases from 4 to 16, which implies, as the distribution becomes narrower (concentrated near zero), the reconstruction performance improves, the best being for $Q = 16$. Surprisingly, $Q = 16$ beats the zero-phase initialization with a 2-dB improvement. In order to investigate the behavior beyond $Q = 16$, we consider phase drawn from the distribution with parameter $Q = 32$. Fig. 4(c), shows that the performance with $Q = 32$ approximates the zero-phase reconstruction and does not improve over that obtained using $Q = 16$.

V. Conclusions

We considered the problem of phase retrieval from autocorrelation measurements, which was cast into a blind deconvolution framework. Due to non-convexity of the formulation, we proposed an alternating minimization strategy, which gave rise to simpler convex sub-problems. The ambiguity in convergence to local and global minima was solved by resorting to a specific initial estimate called zero-phase initialization. Zero-phase estimate for the initialization proved fruitful in arriving at one of the global minima. We considered the class of CDD signals and showed exact phase retrieval using the proposed algorithm based on zero-phase initialization. We also considered random-phase initialization, specifically for the reconstruction of CDD sequences, where the phase was chosen from a uniform distribution concentrated about the origin. We showed that exact reconstruction was possible with phase sequence drawn from highly concentrated distributions. We also showed empirically the existence of a random initialization, which performs better than the zero-phase initialization.

While the deconvolution perspective brings in new insights into the phase retrieval problem, it also brings up several interesting questions. For instance, the role of initialization seems to be extremely crucial and designing a robust initialization scheme that guarantees convergence to the ground-truth

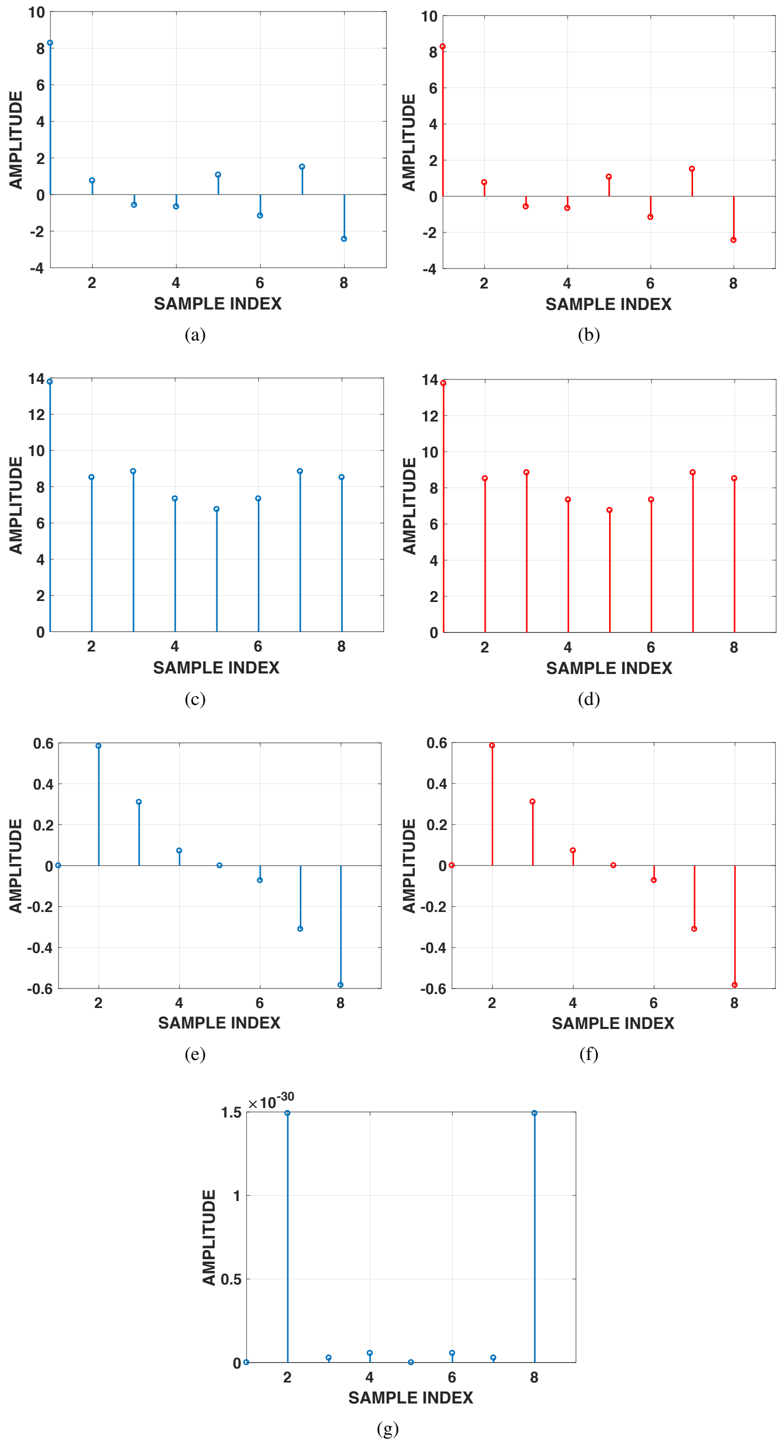

Fig. 3: Demonstration of exact reconstruction for a CDD sequence using zero-phase initialization: (a) Original sequence; (b) Reconstructed sequence; (c) Original DFT magnitude; (d) Reconstructed DFT magnitude; (e) Original DFT phase; (f) Reconstructed DFT phase; and (g) Phase error.

signal is a topic for further research. Applying the proposed algorithm to real-data will be considered separately. Further, one could extend the proposed formalism to sparse signals and modify the cost to include sparsity promoting regularizers –

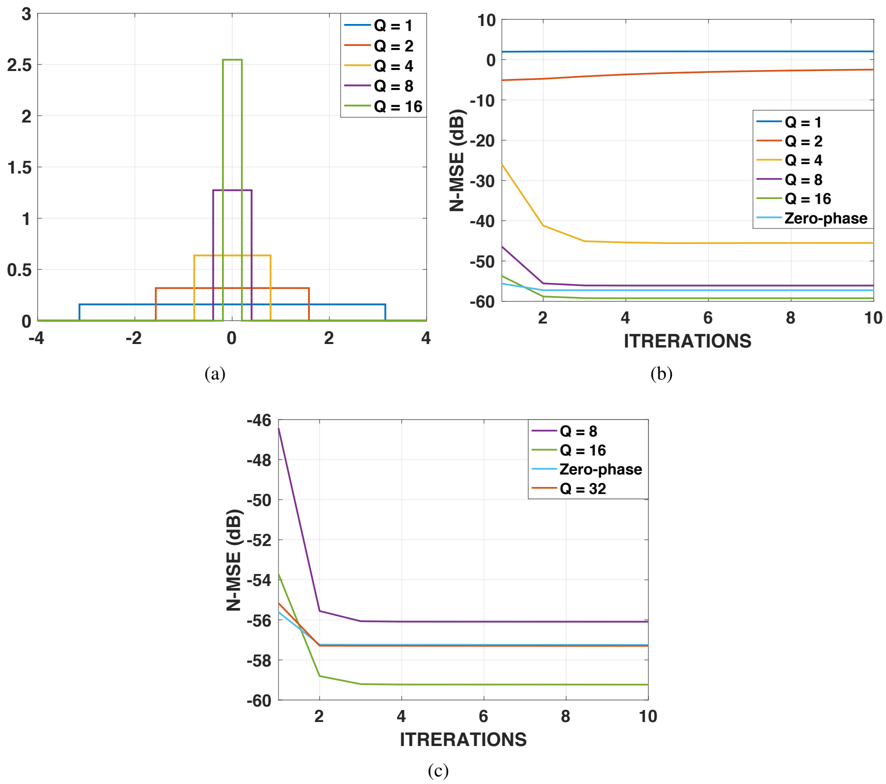

Fig. 4: Comparison of reconstruction performance for the CDD sequence using a random-phase sequence, drawn from a uniform distribution, for the initial estimate. (a) Uniform distributions parameterized by Q; (b) Reconstruction performance corresponding to the distributions in Fig. 4(a) and zero-phase initialization; and (c) Reconstruction performance for Q beyond 16 and its relation to zero-phase initialization.

these are all aspects that warrant further investigation.

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